

Vanishing of the four-loop charge renormalization function in $\mathcal{N} = 4$ SYM theory

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Abstract

We calculate the renormalization constants of the maximally extended $\mathcal{N} = 4$ supersymmetric Yang-Mills theories in the dimensional reduction scheme up to four loops. We have found, that the beta-function is zero both from gauge and Yukawa vertices.

30-years ago the three-loop calculations of the β -function in maximally extended $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory were performed [1, 2, 3]. The result of calculations was zero as in one- and two-loop orders [4, 5]. Generalization to all loop orders was performed further in Refs. [6, 7]. A distinctive feature of these calculations was the necessity of a regularization that preserve supersymmetry. The calculations in Refs. [2, 3] were performed in superfield formalism, while in Ref. [1] component approach was used. A calculation in components is conventional and can be easily done if one already has appropriate method for calculations in usual gauge theories, such as Quantum Chromodynamics (QCD). Namely in this way the calculations of the three-loop β -function in QCD [8] were extended to $\mathcal{N}=4$ SYM theory in Ref. [1]. However, the dimensional regularization, which was used for multi-loop calculations in QCD, violate supersymmetry because in supersymmetric theories one should keep the number of components of all spinors fixed. To restore supersymmetry one should add to the $4 - 2\epsilon$ gauge fields 2ϵ scalar fields [9, 10]. So, in the calculation in $\mathcal{N} = 4$ SYM theory the Dimensional Reduction (DR) scheme prescribes to work with Dirac matrices in four dimensions, while the number of scalar and pseudoscalar fields should be equal $3 + \epsilon$ rather than 3. In this way the vanishing three-loop β -function was obtained both from gauge and Yukawa vertices [1]. However, DR scheme contains internal contradictions [11], which lead to the incorrect result in the higher-loop orders. Such example was found with the generalization of three-loop calculations to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM theories [12]. It was found, that the β -functions of the gauge and Yukawa vertices do not coincide starting from three loops¹. Then, the investigations of applicability of DR scheme were performed [15, 12, 16] and estimations give, that for the $\mathcal{N} = 4$ SYM theory DR scheme should work up to five loops for the propagator type diagrams (see Table 1 in Ref. [12]).

The aim of the work presented in this paper is the calculation of the β -function in $\mathcal{N} = 4$ SYM theory from the gauge and Yukawa vertices in the framework of DR scheme to check the correctness of this scheme in the fourth order of perturbation theory. Moreover, our result for the four-loop renormalization constants can be used for the possible calculations at five-loop order in $\mathcal{N} = 4$ SYM theory, such as calculation of anomalous dimension of Konishi operator [17]² for the testing of integrability in the framework of AdS/CFT-correspondence.

Renormalization constants within MS-like schemes do not depend on dimensional parameters (masses, momenta) [25] and have the following structure:

$$Z = 1 + \sum_{n=1}^{\infty} z^{(n)}(\alpha, g^2) \epsilon^{-n}, \quad (1)$$

where α is the gauge fixing parameter. The renormalization constants define corresponding

¹Despite the fact that the recent calculations give the same β -function from the gauge and Yukawa vertices in the case of $\mathcal{N} = 1$ SYM theory [13, 14], DR scheme still gives the different results in the case of $\mathcal{N} = 2$ SYM theory [14]

²In spite of anomalous dimensions of the twist-2 operators (including Konishi) are known at five-loop order [18, 19] (and at six-loop order for the most simple twist-3 operators [20]), direct calculations are performed only at four loops [21, 22, 23] (for twist-3 at five loops [24]).

anomalous dimensions:

$$\gamma(\alpha, g^2) = g^2 \frac{\partial}{\partial g^2} z^{(1)}(\alpha, g^2) = \sum_{n=1}^{\infty} g^{2(n+1)} \gamma^{(n)}. \quad (2)$$

The renormalization of the coupling constant is related with the renormalization constant of the corresponding vertex and the renormalization constants of the fields entering into this vertex. For the triple vertices we have

$$Z_{g^2} = Z_{jjk}^2 Z_j^{-2} Z_k^{-1}, \quad (3)$$

where Z_{jjk} and Z_j are the renormalization constants for the triple vertices and the wave functions correspondingly. From the last equation one obtains the charge renormalization β -function as

$$\beta_{jjk}(g^2) = g^2 [2\gamma_{jjk}(\alpha, g^2) - 2\gamma_j(\alpha, g^2) - \gamma_k(\alpha, g^2)]. \quad (4)$$

We have calculated the renormalization constants for all fields and for the ghost-ghost-gluon and fermion-fermion-scalar vertices, that give us β -function from the two different type of vertices.

The calculations of the renormalization constants within MS-like scheme can be reduced to the calculation only of massless propagator type diagrams by means of the method of infrared rearrangement [26]. In the case of the gauge (ghost-ghost-gluon in our case) or Yukawa (fermion-fermion-scalar) vertices it means that we can nullify the momentum of the external gauge or scalar fields, correspondingly, reducing the calculation of the Z_{jjk} to the propagator type diagrams.

Calculations of the renormalization constants were made with our program BAMBA [27] based on the algorithm of Laporta [28] (see also [29, 30, 31]), which we used in our previous calculations [23, 32]. All calculations were performed with FORM [33], using FORM package COLOR [34] for evaluation of the color traces and with the Feynman rules from Ref. [35]. For the dealing with a huge number of diagrams we use a program DIANA [36], which call QGRAF [37] to generate all diagrams enumerated in Table 1.

	1-loop	2-loop	3-loop	4-loop
Ghost wave function	1	8	158	4.563
Scalar wave function	2	34	930	37.014
Fermion wave function	3	40	1.210	51.465
Gluon wave function	5	58	1.513	57.664
Ghost-ghost-gluon vertex	2	47	1.462	57.939
Fermion-fermion-scalar vertex	5	183	8.845	517.576
Sum	18	370	14.118	726.221

Table 1: The number of diagrams for calculations up to four-loop order.

The results of calculations up to four-loop order are the following:

$$\gamma_3 = -2 C_A a + C_A^2 a^2 + \left(-\frac{59}{16} - \frac{63}{4} \zeta_3 \right) C_A^3 a^3 + \left(\left(-\frac{305}{192} - \frac{16325}{96} \zeta_5 + \frac{45}{16} \zeta_4 + \frac{2797}{96} \zeta_3 \right) C_A^4 + \left(-9 + \frac{125}{4} \zeta_5 + \frac{185}{4} \zeta_3 \right) d_{44} \right) a^4, \quad (5)$$

$$\tilde{\gamma}_3 = \frac{1}{2} C_A a - \frac{5}{4} C_A^2 a^2 + \left(\frac{155}{32} + \frac{63}{8} \zeta_3 \right) C_A^3 a^3 + \left(\left(\frac{5849}{384} + \frac{14725}{192} \zeta_5 + \frac{81}{32} \zeta_4 + \frac{499}{192} \zeta_3 \right) C_A^4 + \left(\frac{9}{2} - \frac{265}{8} \zeta_5 - \frac{49}{8} \zeta_3 \right) d_{44} \right) a^4, \quad (6)$$

$$\tilde{\gamma}_1 = -\frac{1}{2} C_A a - \frac{3}{4} C_A^2 a^2 + 3 C_A^3 a^3 + \left(\left(\frac{231}{16} - \frac{25}{3} \zeta_5 + \frac{63}{16} \zeta_4 + \frac{103}{6} \zeta_3 \right) C_A^4 + \left(-\frac{35}{2} \zeta_5 + 17 \zeta_3 \right) d_{44} \right) a^4, \quad (7)$$

$$\gamma_\phi = -2 C_A a - C_A^2 a^2 + \left(\frac{23}{4} - \frac{27}{2} \zeta_3 \right) C_A^3 a^3 + \left(\left(\frac{561}{16} - \frac{7145}{48} \zeta_5 + \frac{57}{8} \zeta_4 + \frac{2597}{48} \zeta_3 \right) C_A^4 + \left(-7 + \frac{185}{2} \zeta_5 - \frac{59}{2} \zeta_3 \right) d_{44} \right) a^4, \quad (8)$$

$$\gamma_\lambda = -4 C_A + 6 C_A^2 a^2 + \left(-\frac{101}{4} - 27 \zeta_3 \right) C_A^3 a^3 + \left(\left(-\frac{5591}{48} - \frac{3185}{12} \zeta_5 + \frac{51}{8} \zeta_4 - \frac{1009}{24} \zeta_3 \right) C_A^4 + (-8 - 320 \zeta_5 + 142 \zeta_3) d_{44} \right) a^4, \quad (9)$$

$$\gamma_4 = -5 C_A a + \frac{11}{2} C_A^2 a^2 + \left(-\frac{179}{8} - \frac{135}{4} \zeta_3 \right) C_A^3 a^3 - \left(\left(\frac{9499}{96} + \frac{10875}{32} \zeta_5 - \frac{159}{16} \zeta_4 + \frac{1439}{96} \zeta_3 \right) C_A^4 + \left(\frac{23}{2} + \frac{1095}{4} \zeta_5 - \frac{509}{4} \zeta_3 \right) d_{44} \right) a^4, \quad (10)$$

with $a = g^2/(4\pi)^2$ and the following Casimir operators of gauge group $SU(N)$: $C_A = N$, $d_{44} = N^2(N^2 + 36)/24$. Here $\tilde{\gamma}_1$ and γ_4 are the anomalous dimensions of the ghost-ghost-gluon and fermion-fermion-scalar vertices, and γ_3 , $\tilde{\gamma}_3$, γ_ϕ and γ_λ are those of gluon, ghost, scalar, and fermion fields, respectively. The three-loop results for the same quantities can be found in Ref. [1].

Substituting the obtained γ -functions into Eq. (4) we have found both from the ghost-ghost-gluon and fermion-fermion-scalar that the β -function is equal to zero:

$$\beta^{4\text{-loop}}(a) = 0. \quad (11)$$

So, we have found, that the gauge and Yukawa couplings are renormalized in the same way in $\mathcal{N} = 4$ SYM theory. Hence, the DR-scheme preserves supersymmetry and works correctly in these model up to four loops. To conclude, we note again that the obtained renormalization constants can be used for the possible calculation of the five-loop anomalous dimension of Konishi operator in $\mathcal{N} = 4$ SYM theory.

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